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Alternative factorization of eigenvalue problems in one dimension

F M Fernández†, A Lopez Piñeiro and B Moreno
Universidad de Extremadura, Departamento de Química Física, 06071-Badajoz, Spain

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Abstract. We propose an alternative factorization of eigenvalue problems in one dimension. The method is based on simple equations connecting a pair of solutions to two second-order differential equations that differ in the coefficient of the independent variable. Under certain conditions these connection equations play the role of recurrence relations. The method is particularly suitable for the treatment of separable quantum-mechanical problems giving rise to a consistency condition which tells us whether a potential is shape-invariant. From this consistency condition we derive a simple algorithm for the construction of partner potentials and shape-invariant potentials. The present connection method appears to be more general than both the standard factorization method and supersymmetric quantum mechanics. As illustrative examples we consider Bessel's and Legendre's equations, the generalized Kepler problem, inverse quadratic potentials and an asymmetric potential well.

1. Introduction

Recently, Fernández [1, 2], and Fernández and Castro [3, 4] developed a new approach to perturbation theory for separable quantum-mechanical problems which consists of writing the perturbed states Ψ in terms of the unperturbed one Φ and its first derivative Φ' according to $\Psi(x) = A(x)\Phi(x) + B(x)\Phi'(x)$. The main advantage of this method is that Φ does not appear explicitly in the coupled differential equations for the unknown functions $A(x)$ and $B(x)$. For this reason the results of this procedure apply to an arbitrary state of the system, allowing one to express the perturbed eigenvalue and eigenfunction in terms of the quantum numbers of the unperturbed problem.

If instead of being solutions of unperturbed and perturbed problems the functions Φ and Ψ are two solutions of the same problem the connection method proposed by Fernández and Castro [4] resembles, when $B'(x) \equiv 0$, the factorization method extensively investigated by Infeld and Hull [5]. However, the points of view of both approaches differ in that the latter attempts a factorization of a differential operator and the former focuses on the connection between two solutions. Although equivalent to both the factorization method and to supersymmetric quantum mechanics [6–8] when $B' = 0$, the connection method may be more general than the former approaches as it allows connection formulae with $B' \neq 0$. The purpose of this paper is to investigate the additional flexibility provided by this function.

In section 2 we develop general equations which apply to most exactly solvable quantum-mechanical problems and compare the connection method with the factorization

† Permanent address: QUINOR, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Calle 47 y 115, Casilla de Correo 962, 1900 La Plata, Argentina.

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method and its generalization by means of supersymmetric quantum mechanics [5–8]. In section 3 we derive recurrence relations for integrals which appear in the equation connecting the norms of a pair of solutions. In section 4 we obtain recurrence relations for Bessel's functions. In section 5 we treat Legendre's equation. In section 6 we consider the variation of two parameters in the quantum-mechanical eigenvalue equation for a generalized Kepler problem which are related to the angular and principal quantum numbers in the particular case of the Coulomb interaction. In section 7 we consider the eigenvalue equation for an arbitrary inverse quadratic potential from which one derives several exactly solvable problems in quantum mechanics. Among them we mention the associated spherical harmonics, the generalized spherical harmonics and the quantum-mechanical symmetric top. In section 8 we develop a simple algorithm for the systematic construction of partner and shape-invariant potentials. Further comments and conclusions are found in section 9.

2. The method

It is our purpose to connect a solution $Y(x)$ to the ordinary differential equation

$$P(x)Y''(x) + Q(x)Y'(x) + \bar{R}(x)Y(x) = 0 \tag{1}$$

with a solution $Z(x)$ to a closely related equation

$$P(x)Z''(x) + Q(x)Z'(x) + \bar{R}_0(x)Z(x) = 0. \tag{2}$$

Here, the prime denotes differentiation with respect to x and the coefficients $P(x)$, $Q(x)$, $\bar{R}(x)$ and $\bar{R}_0(x)$ are real differentiable functions of the independent variable x in the interval (x_1, x_2). Although it is not strictly necessary, for simplicity we remove the first derivative in each equation by means of the transformations

$$Y(x) = F(x)\Psi(x) \qquad Z(x) = F(x)\Phi(x). \tag{3}$$

Choosing

$$F(x) = \exp\left\{-\int^x Q(t)/[2P(t)]\,\mathrm{d}t\right\} \tag{4}$$

the resulting differential equations for $\Psi(x)$ and $\Phi(x)$ are

$$P(x)\Psi''(x) + R(x)\Psi(x) = 0 \tag{5}$$

$$P(x)\Phi''(x) + R_0(x)\Phi(x) = 0 \tag{6}$$

where

$$R(x) = \bar{R}(x) - \frac{Q(x)^2}{4P(x)} - \frac{P(x)}{2}\left(\frac{Q(x)}{P(x)}\right)' \tag{7}$$

and similarly for $R_0(x)$.

In this paper we try the following connection between $\Psi(x)$ and $\Phi(x)$:

$$\Psi(x) = A(x)\Phi(x) + \bar{B}(x)\Phi'(x) \tag{8}$$

suggested by recent applications of perturbation theory to separable quantum-mechanical problems [1–4]. The functions $A(x)$ and $\bar{B}(x)$ are determined by the condition that (8) be a solution of (5). Substitution of (8) into (5) followed by elimination of the second and third derivatives of $\Phi(x)$ by means of (6) leads to an equation for $\Phi(x)$ and $\Phi'(x)$. Choosing $A(x)$ and $\bar{B}(x)$ so that the coefficients of the former functions vanish, we obtain the coupled ordinary differential equations

$$PA'' + (R - R_0)A - P\left(\frac{R_0\bar{B}}{P}\right)' - R_0\bar{B}' = 0 \tag{9}$$

$$\bar{B}'' + (R - R_0)\frac{\bar{B}}{P} + 2A' = 0. \tag{10}$$

In order to remove $P(x)$ from the denominators in (9) and (10) we define the new function $B(x) = \bar{B}(x)/P(x)$ so that those equations become

$$PA'' + (R - R_0)A - 2PR_0B' - (PR_0)'B = 0 \tag{11}$$

$$PB'' + 2P'B' + P''B + (R - R_0)B + 2A' = 0. \tag{12}$$

If there exist two differentiable functions $A(x)$ and $B(x)$ satisfying (11) and (12), then we derive a solution to (5) from a given solution to (6) according to (8). Moreover, the corresponding solutions to the original equations (1) and (2) are related by

$$Y(x) = \left(A(x) + \frac{B(x)Q(x)}{2}\right) Z(x) + P(x)B(x)Z'(x). \tag{13}$$

To obtain the inverse transformation to (8) we solve the linear system of two equations formed by (8) and its first derivative $\Psi' = (A' - R_0\bar{B}/P)\Phi + (A + \bar{B}')\Phi'$ for Φ and Φ'. The result is

$$\Phi(x) = A_0(x)\Psi(x) + \bar{B}_0(x)\Psi'(x) \tag{14}$$

where

$$A_0(x) = \frac{A(x) + \bar{B}'(x)}{C} \qquad \bar{B}_0(x) = -\frac{\bar{B}(x)}{C} \tag{15}$$

and

$$C = A(x)^2 + A(x)\bar{B}'(x) - A'(x)\bar{B}(x) + \frac{R_0(x)\bar{B}(x)^2}{P(x)}. \tag{16}$$

Subtracting A times (10) from $\bar{B}$ times (9), one easily proves that C is independent of x. The inverse transformation exists provided that $C \neq 0$. To derive $Z(x)$ from $Y(x)$ we simply interchange these functions in (13) and substitute $A_0(x)$ and $B_0(x)$ for $A(x)$ and $B(x)$, respectively.

Assuming that the functions Ψ and Φ vanish at the end-points of the interval (x_1, x_2) the connection formula (8) also allows us to relate the integrals of Ψ^2 and Φ^2 over that interval. A straightforward integration by parts shows that

$$\int_{x_1}^{x_2} \Psi(x)^2\,\mathrm{d}x = \int_{x_1}^{x_2} \left\{C - 2A(x)\bar{B}'(x) + \frac{[\bar{B}(x)^2]''}{2}\right\} \Phi(x)^2\,\mathrm{d}x. \tag{17}$$

As shown in appendix 1, it is always possible to transform (1) and (2) into equivalent second-order differential equations with $P = 1$, $Q = 0$ and modified coefficients R and R_0. In some of such cases an appropriate change of the independent and dependent variables of the differential equation gives rise to connection formulae in which the function $B(x)$ is simply a constant. This is a particularly interesting case because the relevant equations developed above reduce to

$$A'' + (R - R_0)A - R_0'B = 0 \tag{18}$$

$$(R - R_0)B + 2A' = 0. \tag{19}$$

If we differentiate (19) with respect to x, substitute the resulting equation into (18) and then solve for A we have

$$A = B\frac{R' + R_0'}{2(R - R_0)}. \tag{20}$$

From (19) and (20) one easily obtains the following consistency condition:

$$(R - R_0)(R + R_0)'' + R_0'^2 - R'^2 + (R - R_0)^3 = 0. \tag{21}$$

If R and R_0 satisfy this equation then $\bar{B}(x) = B(x) = B =$ constant (remember that $P = 1$) in the connection formula (8). Furthermore, since (17) reduces to

$$\int_{x_1}^{x_2} \Psi(x)^2\,\mathrm{d}x = C\int_{x_1}^{x_2} \Phi(x)^2\,\mathrm{d}x \tag{22}$$

we conclude that $C \geqslant 0$ for square integrable solutions.

The particular case just discussed leads to equations that are identical to those in the factorization method and in supersymmetric quantum mechanics [5–8]. For example, long ago Infeld [9] developed an equation similar to (20). By substitution of (20) into (16) it is easy to obtain an expression similar to that derived by Infeld [5,9] for $L(m)$ which is proportional to our C. Furthermore, the only function in the connection formula $A(x)$ is proportional to the superpotential in supersymmetric quantum mechanics [7,8]. In other words, both the factorization method and supersymmetric quantum mechanics appear to be a particular case of the connection method when $P = 1$, $Q = 0$ and $B' = 0$.

3. Recurrence relations for integrals

In what follows we develop a recurrence relation that greatly facilitates the calculation of the integrals which commonly appear in the right-hand side of (17). To this end we first multiply (2) by $f(x)Z'(x)$, where $f(x)$ is an arbitrary well behaved function, and integrate the resulting equation between x_1 and x_2. Rearranging the first two terms of the integrand as $fPZ'Z'' = \frac{1}{2}(fPZ'^2)' - \frac{1}{2}(fP)'Z'^2$ and $f\bar{R}_0ZZ' = \frac{1}{2}(f\bar{R}_0Z^2)' - \frac{1}{2}(f\bar{R}_0)'Z^2$, and assuming that both fPZ'^2 and $f\bar{R}_0Z^2$ vanish at the end-points, we obtain

$$\int_{x_1}^{x_2}\left\{\frac{(fP)' - 2fQ}{2}Z'^2 + \frac{(f\bar{R}_0)'}{2}Z^2\right\}\mathrm{d}x = 0. \tag{23}$$

We next multiply (2) by $g(x)Z(x)$, where $g(x)$ is another well behaved function, and operate exactly in the same way to obtain

$$\int_{x_1}^{x_2} \left\{ gPZ'^2 - \frac{(gP)''}{2}Z^2 + \frac{(gQ)'}{2}Z^2 - g\bar{R}_0 Z^2 \right\} \mathrm{d}x = 0. \tag{24}$$

Choosing $g = [fQ - (fP)'/2]/P$ we eliminate the terms containing Z'^2 between the two equations above. The resulting expression

$$\int_{x_1}^{x_2} \left\{ \frac{(fP)'''}{2} - (fQ)' + \left(\frac{fQ^2}{P} - \frac{(fP)'Q}{2P} \right)' + \frac{(fP)'\bar{R}_0}{P} - \frac{2fQ\bar{R}_0}{P} + (f\bar{R}_0)' \right\} Z^2 \, \mathrm{d}x = 0 \tag{25}$$

allows us to calculate the integrals that are necessary to compare the norms of the functions Y and Z. In particular, when $P = 1$ and $Q = 0$ then $\bar{R}_0 = R_0$, $Z = \Phi$, and (25) reduces to

$$\int_{x_1}^{x_2} \left\{ \frac{f'''}{2} + 2f'R_0 + fR_0' \right\} \Phi^2 \, \mathrm{d}x = 0. \tag{26}$$

Equation (25) is a generalization of the well known diagonal hypervirial relations [10].

In the remainder of this paper we apply the general results just obtained to some selected examples in mathematical physics and quantum mechanics.

4. Bessel's equation

Bessel's equation

$$x^2 Y''(x) + xY'(x) + (x^2 - n^2)Y(x) = 0 \tag{27}$$

is a particular case of (1) with $P(x) = x^2$, $Q(x) = x$, and $\bar{R}(x) = x^2 - n^2$, where n is a real number. We write $\bar{R}_0(x) = x^2 - m^2$ so that after elimination of xY' as discussed before we have $R(x) = \bar{R}(x) + \frac{1}{4}$ and $R_0(x) = \bar{R}_0(x) + \frac{1}{4}$. It is not difficult to verify that for this example the polynomials

$$A(x) = \sum_j a_j x^j \qquad B(x) = \sum_j b_j x^j \tag{28}$$

are solutions to (11) and (12), provided that the coefficients satisfy the recurrence relations

$$a_j = \frac{n^2 - m^2 - j(j+1)}{2j} b_{j-1} \qquad j \neq 0 \qquad b_{-1} = 0 \tag{29}$$

$$u_j b_j + v_j b_{j-2} = 0 \tag{30}$$

where

$$\begin{aligned} u_j &= \frac{(j+1)(j+2) + m^2 - n^2}{2}[j(j+1) + m^2 - n^2] - (j+1)^2(2m^2 - \tfrac{1}{2}) \\ v_j &= 2j(j+1). \end{aligned} \tag{31}$$

For simplicity we look for the minimal polynomial solution, i.e. the polynomial with the least number of terms. It follows from (29) and (30) that $b_{2i+1} = 0$ for all values of i. Furthermore, setting $b_{k-2} = b_{k+2} = 0$ and $b_k \neq 0$ assures us that $b_j = 0$ for all $j \neq k$. This last condition requires that $v_{k+2} = u_k = 0$, which is possible provided that $k = -1$ or $k = -2$. The former leads to the trivial solution $m = \pm n$ and the latter to a non-trivial relationship between m and n:

$$(m^2 - n^2)^2 + 2(m^2 - n^2) + 1 - 4m^2 = 0. \tag{32}$$

The two roots of this quadratic equation are consistent with $n = m + \sigma$, where $\sigma^2 = 1$. One easily verifies that polynomial solutions of larger order connect Bessel's functions with $|n - m| > 1$. Such larger polynomials also arise from repeated application of the connection formula (8) with the minimal polynomials followed by the use of the differential equation (6) to eliminate second and higher derivatives of Φ.

The explicit expressions for the minimal polynomials $A(x)$ and $B(x)$ just obtained are

$$A(x) = -\frac{b(2m\sigma + 1)}{2x} \qquad B(x) = \frac{b}{x^2} \qquad \bar{B}(x) = b \tag{33}$$

where we have written b instead of b_{-2}, which depends on n, to simplify the notation. Although the functions $A(x)$ and $B(x)$ in (33) apply to both cases $n = m \pm 1$, it is convenient to choose one of them and derive the inverse transformation according to (14)–(16) using the same value of b. This procedure gives

$$C = b^2 \qquad A_0(x) = -\frac{2m\sigma + 1}{2bx} \qquad \bar{B}_0 = -\frac{1}{b}. \tag{34}$$

Taking into account (13) and the results just obtained, and writing Y_n and $Y_m = Y_{n-1}$ instead of Y and Z, respectively, we conclude that

$$Y_n = b\left(\frac{1-n}{x} + \frac{\mathrm{d}}{\mathrm{d}x}\right) Y_{n-1} \qquad Y_{n-1} = -\frac{1}{b}\left(\frac{n}{x} + \frac{\mathrm{d}}{\mathrm{d}x}\right) Y_n \tag{35}$$

which agree with standard recurrence relations for Bessel's functions if $b = -1$ [11].

Since $\bar{B}$ is constant we can try the simplified version of the connection method. Dividing (27) by x^2, the resulting equation is of the form (1) with $P = 1$, $Q = 1/x$ and $\bar{R} = (x^2 - n^2)/x^2$. Elimination of the terms Y'/x and Z'/x leaves $R = 1 + (1 - 4n^2)/4x^2$ and $R_0 = 1 + (1 - 4m^2)/4x^2$. These functions satisfy (21), provided that one of the following equations is satisfied: $m^2 = n^2$, $(m + n)^2 = 1$ and $(m - n)^2 = 1$. The first one leads to trivial results and the remaining two reflect the fact that Bessel's equation is invariant under the change of the sign of n (or m). Therefore, it is sufficient to consider the condition $n = m + \sigma$. Since (20) and (16) yield $A(x) = -B(2m + \sigma)/2\sigma x$ and $C = B^2$, respectively, we recover the results obtained above from the minimal polynomial solutions.

5. Legendre's equation

Legendre's equation

$$(1 - x^2)Y''(x) - 2xY'(x) + l(l + 1)Y(x) = 0 \tag{36}$$

where l is a real constant, is also of the form (1) with $P(x) = 1 - x^2$, $Q(x) = -2x$ and $\bar{R}(x) = \lambda = l(l+1)$. It is our purpose to relate a solution to (36) with a solution to the same equation with $\bar{R}_0(x) = \lambda' = l'(l'+1)$. After elimination of the term $2xY'$ according to (3) and (4) we are left with an equation like (5) with

$$R(x) = \lambda + \frac{1}{1-x^2} \tag{37}$$

and a similar expression for $R_0(x)$.

A straightforward calculation shows that the coefficients of the polynomials (28) that are solutions to the differential equations (11) and (12) for this example satisfy the recurrence relations

$$a_j = -\frac{j+1}{2} b_{j+1} - \frac{\lambda - \lambda' - j(j+1)}{2j} b_{j-1} \qquad j \neq 0 \tag{38}$$

$$u_j b_{j+2} + v_j b_j + w_j b_{j-2} = 0 \tag{39}$$

where

$$\begin{aligned} u_j &= \frac{j(j^2-1)(j+2)}{2} \qquad v_j = j(j-1)(\lambda + \lambda' - j^2) \\ w_j &= [\lambda - \lambda' - (j-1)(j-2)]\frac{\lambda - \lambda' - j(j-1)}{2} - 2\lambda'(j-1)^2. \end{aligned} \tag{40}$$

In order to obtain a minimal solution we set $b_{2i+1} = 0$ for all i, $b_{k-2} = b_{k-4} = b_{k+2} = b_{k+4} = 0$ and $b_k \neq 0$, which is possible provided that either $k = 0$ and $w_0 = 0$ or $k = 1$ and $w_1 = 0$. The latter condition leads to the trivial result $\lambda = \lambda'$, and the former determines the following non-trivial relationship between λ and λ':

$$(\lambda - \lambda')^2 - 2(\lambda - \lambda') - 4\lambda' = 0. \tag{41}$$

The two roots of this quadratic equation are equivalent to $l = l' \pm 1$, and without loss of generality we choose $l = l' + 1$. As in the preceding example, polynomial solutions of larger order connect solutions to Legendre's equation with $|l - l'| > 1$.

The minimal polynomial solutions are

$$A(x) = (1-l)Bx \qquad B(x) = B = \text{constant} \tag{42}$$

and for the inverse transformation we have

$$A_0(x) = -\frac{l+1}{Bl^2}x \qquad \bar{B}_0(x) = -\frac{1-x^2}{Bl^2} \tag{43}$$

because $C = B^2 l^2$. These results determine the recurrence relations

$$\begin{aligned} Y_l &= b\left(-lx + (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right) Y_{l-1} \\ Y_{l-1} &= -\frac{1}{Bl^2}\left(lx + (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right) Y_l \end{aligned} \tag{44}$$

after transforming back from Ψ and Φ to Y and Z, respectively, and setting $Y = Y_l$ and $Z = Y_{l-1}$. When l is a positive integer and $B = -1/l$ these equations become well known recurrence relations for the Legendre polynomials $P_l(x)$ which satisfy $P_l(1) = 1$ [11].

6. The generalized Kepler problem

The radial part of the stationary Schrödinger equation for the generalized Kepler problem (or Kratzer potential) is of the form (atomic units are used throughout)

$$\Psi''(r) + \left(2E + \frac{2Z}{r} - \frac{\lambda}{r^2}\right)\Psi(r) = 0 \tag{45}$$

where E is the energy and Z and λ are real constants. The physical solutions satisfy $\Psi(0) = 0$ and in the case of bound states ($Z > 0, E < 0$) also $\Psi(r \to \infty) = 0$. The parameter λ may depend on a potential parameter, on the angular momentum quantum number, and also on the dimension of the space. However, for our present purposes E, Z and λ are merely real parameters and we disregard their physical meaning.

We first try to connect solutions to (45) corresponding to different values of λ and accordingly choose $P(r) \equiv 1$, $Q(r) \equiv 0$ and

$$R(r) = 2E + \frac{2Z}{r} - \frac{\lambda}{r^2} \qquad R_0(r) = 2E + \frac{2Z}{r} - \frac{\lambda'}{r^2}. \tag{46}$$

These functions satisfy (21) provided that $\lambda - \lambda' = 1 \pm \sqrt{1 + 4\lambda'}$, which requires that $\lambda' \geqslant -\frac{1}{4}$ (and similarly for λ). Substituting $\lambda = \lambda' + 2\xi$ and solving for λ' we obtain $\lambda' = \xi(\xi - 1)$ and $\lambda = \xi(\xi + 1)$. Shape invariance demands that we define ξ' according to $\lambda' = \xi'(\xi' + 1)$, from which it follows that $\xi' = \xi - 1$ or $\xi' = -\xi - 1$. The parameters ξ and ξ' are bounded from below, $\xi, \xi' \geqslant \xi_m \geqslant -\frac{1}{2}$ to be consistent with $\lambda, \lambda' \geqslant -\frac{1}{4}$. Under such conditions $B(r) = B =$ constant, and (20) reads

$$A(r) = B\left(\frac{Z}{\xi} - \frac{\xi}{r}\right). \tag{47}$$

The inverse transformation is given by

$$A_0(r) = \frac{A(r)}{C(\xi)} \qquad B_0 = -\frac{B}{C(\xi)} \tag{48}$$

where

$$C(\xi) = B^2\left(\frac{Z^2}{\xi^2} + 2E\right). \tag{49}$$

The results just obtained lead to the following recurrence relations or ladder equations:

$$\Psi_\xi = B\left\{\left(\frac{Z}{\xi} - \frac{\xi}{r}\right)\Psi_{\xi-1} + \Psi'_{\xi-1}\right\} \tag{50}$$

$$\Psi_{\xi-1} = \frac{B}{C(\xi)}\left\{\left(\frac{Z}{\xi} - \frac{\xi}{r}\right)\Psi_\xi - \Psi'_\xi\right\}. \tag{51}$$

These equations apply to both discrete and continuous spectra because we have not yet specified the values of the parameters in the differential equation (45) except for the restriction upon λ and λ'. In the case of a discrete spectrum we have to find the conditions under which the solutions are square integrable. Writing equation (22) as

$$\int_0^\infty \Psi_\xi^2 \,\mathrm{d}r = C(\xi) \int_0^\infty \Psi_{\xi-1}^2 \,\mathrm{d}r \tag{52}$$

we conclude that if $\Psi_{\xi-1}$ is square integrable and $C(\xi) > 0$ then Ψ_ξ will also be square integrable. As long as $\xi^2 \leqslant -Z^2/2E$ we can choose the arbitrary constant B to be

$$B = \frac{1}{\sigma(\xi)} \qquad \sigma(\xi) = \left(\frac{Z^2}{\xi^2} + 2E\right)^{1/2} \tag{53}$$

so that the norms of Ψ_ξ and $\Psi_{\xi-1}$ are the same. Equations (50) and (51) thus become

$$\left(\frac{Z}{\xi+1} - \frac{\xi+1}{r}\right)\Psi_\xi + \Psi_\xi' = \sigma(\xi+1)\Psi_{\xi+1} \tag{54}$$

$$\left(\frac{Z}{\xi} - \frac{\xi}{r}\right)\Psi_\xi - \Psi_\xi' = \sigma(\xi)\Psi_{\xi-1}. \tag{55}$$

The transformation (54) increases ξ by unity so that $\sigma(\xi)^2$ would eventually become negative unless $\sigma(\xi_M + 1) = 0$. Setting this condition in (54) enables one to obtain a square integrable function Ψ_{ξ_M} from which with the help of (55) we derive all the other square integrable functions for the same value of $E < 0$. The possible values of ξ are therefore $\xi_M, \xi_M - 1, \ldots, \xi_M - \nu = \xi_m$, where ν is a positive integer. The value of the energy for the bound states connected by the recurrence relations (54) and (55) is therefore given by the well known expression

$$E = -\frac{Z^2}{2(\xi_M+1)^2} = -\frac{Z^2}{2(\xi_m+\nu+1)^2}. \tag{56}$$

In order to connect states with different energy one may try $R_0(r) = 2E' + 2Z/r - \lambda/r^2$, but this choice does not lead to polynomial solutions to (11) and (12). To overcome this difficulty we shift the energy dependence to another term of $R(r)$ by means of the change of coordinate $x = r/\gamma$, $\gamma = Z/\sqrt{-2E}$, which enables us to rewrite the Schrödinger equation as

$$x^2\bar{\Psi}''(x) + (-Z^2x^2 + 2\gamma Zx - \lambda)\bar{\Psi}(x) = 0 \tag{57}$$

where $\bar{\Psi}(x) = \Psi(\gamma x)$. We now choose $P(x) = x^2$, $Q(x) \equiv 0$,

$$R(x) = -Z^2x^2 + 2\gamma Zx - \lambda \qquad R_0(x) = -Z^2x^2 + 2\gamma' Zx - \lambda \tag{58}$$

forgetting for the moment the dependence of x on E. The coefficients of the polynomial solutions to the differential equations (11) and (12) for this case satisfy

$$\begin{aligned} a_0 &= \frac{\lambda}{Z(\gamma'-\gamma)} b_0 + \frac{\gamma'}{\gamma-\gamma'} b_{-1} \\ a_j &= -\frac{j+1}{2} b_{j-1} - \frac{Z(\gamma-\gamma')}{j} b_{j-2} \qquad j \neq 0 \end{aligned} \tag{59}$$

$$u_j b_j + v_j b_{j-1} + w_j b_{j-2} = 0 \tag{60}$$

where

$$u_j = j(j+1)\left(\frac{j(j+2)}{2} - 2\lambda\right)$$
$$v_j = Zj\left(\frac{(3j+2)(\gamma-\gamma')}{2} + 2\gamma'(2j+1)\right) \tag{61}$$
$$w_j = 2Z^2((\gamma-\gamma')^2 - j^2).$$

One easily verifies that the minimal solution is $b_j = 0$ if $j \neq -1$, which leads to $\gamma = \gamma' \pm 1$,

$$A(x) = \frac{b}{\gamma-\gamma'}(\gamma' - Zx) \qquad B(x) = \frac{b}{x} \tag{62}$$

where we have defined $b = b_{-1}$ to simplify the notation. The inverse transformation is given by

$$A_0(x) = \frac{b(\gamma - Zx)}{C(\gamma-\gamma')} \qquad \bar{B}_0(x) = -\frac{bx}{C} \tag{63}$$

where

$$C = b^2(\gamma\gamma' - \lambda). \tag{64}$$

Without loss of generality we choose $\gamma = \gamma' + 1$ from now on.

In the present case the relation (17) between the norms of the bound states $\bar{\Psi}$ and $\bar{\Phi}$ becomes

$$\int_0^\infty \bar{\Psi}(x)^2\,dx = \int_0^\infty \{C + b^2[2Zx - 2\gamma' + 1]\}\bar{\Phi}(x)^2\,dx. \tag{65}$$

In order to obtain the integral on the right-hand side we make use of (26) with $R_0 = -Z^2 + 2\gamma' Z/x - \lambda/x^2$ and $f(x) = x^N$, $N = 0, 1, \ldots$, which leads to the following recurrence relation:

$$\int_0^\infty \left\{(N-1)\left(\frac{N(N-2)}{2} - 2\lambda\right)x^{N-3} + 2\gamma' Z(2N-1)x^{N-2} - 2NZ^2x^{N-1}\right\}\bar{\Phi}^2\,dx = 0. \tag{66}$$

A straightforward calculation yields

$$\int_0^\infty \bar{\Phi}(x)^2 x\,dx = \frac{3\gamma'^2 - \lambda}{2Z\gamma'}\int_0^\infty \bar{\Phi}(x)^2\,dx \tag{67}$$

so that

$$\int_0^\infty \bar{\Psi}(x)^2\,dx = \frac{C\gamma'}{\gamma}\int_0^\infty \bar{\Phi}(x)^2\,dx. \tag{68}$$

The transformations determined by (62) and (63) connect bound states as long as $\gamma(\gamma-1) \geqslant \lambda$ ($C \geqslant 0$). Therefore, γ is bounded from below by $\gamma_{\min}$, $\gamma \geqslant \gamma_{\min}$, which is related to λ by $\lambda = \gamma_{\min}(\gamma_{\min} - 1)$ thus truncating the downstairs recurrence relation. If we choose

$$b = \frac{\sqrt{\gamma - 1}}{\sqrt{\gamma[\gamma(\gamma-1) - \lambda]}} \tag{69}$$

then the recurrence relations

$$\bar{\Psi}_\gamma = b((\gamma - 1 - Zx)\bar{\Psi}_{\gamma-1} + x\bar{\Psi}'_{\gamma-1}) \tag{70}$$

$$\bar{\Psi}_{\gamma-1} = \frac{b}{C(\gamma)}((\gamma - Zx)\bar{\Psi}_\gamma - x\bar{\Psi}'_\gamma) \tag{71}$$

do not change the norm of the bound states.

7. Inverse quadratic potential

Model potentials of the form $V(x) = \alpha/v(x)^2$, where $v(x)$ has no zeros in (x_1, x_2), exhibit many physical applications. The functions

$$R(x) = E - \frac{\alpha}{v(x)^2} \qquad R_0(x) = E - \frac{\alpha'}{v(x)^2} \tag{72}$$

satisfy (21) provided that

$$v'(x)^2 + v(x)v''(x) = \frac{(\alpha - \alpha')^2}{2(\alpha + \alpha')}. \tag{73}$$

Several functions satisfy (73), and we discuss some of them in what follows.

Although the function $v(x) = x - x_0$ violates the requirement above when $x_1 = -\infty$ and $x_2 = \infty$, we consider it here because recently there has been interest in singular superpotentials, one of which leads to a potential of the form $\alpha/(x - x_0)^2$ [12]. The left-hand side of (73) is unity, and substitution of $\alpha = \alpha' + 2\xi$ shows that $\alpha' = \xi(\xi - 1)$ and $\alpha = \xi(\xi + 1)$. Shape invariance requires that $\alpha' = \xi'(\xi' + 1)$ so that $\xi' = \xi - 1$ or $\xi' = -\xi - 1$ and thus we obtain the partner potentials derived by supersymmetric quantum mechanics [12]. Depending on the value of ξ these partner potentials may or may not share the same Hilbert space [12].

Other functions that satisfy (73) are $\cos(x)$ $(-\pi/2 < x < \pi/2)$, $\cosh(x)$ $(-\infty < x < \infty)$ and $\sin(x)$ $(-\pi < x < \pi)$. They give rise to well known potentials in quantum mechanics. Here we restrict ourselves to $v(x) = \sin(x)$ because it is probably the most useful case in physical applications. Because the left-hand side of (73) is unity as before, the potential parameters are $\alpha = \xi(\xi + 1)$ and $\alpha' = \xi'(\xi' + 1) = \xi(\xi - 1)$ and the eigenvalue equation reads

$$Y_\xi''(x) + \left(E - \frac{\xi(\xi + 1)}{\sin(x)^2}\right) Y_\xi(x) = 0. \tag{74}$$

After obtaining $A(x)$ from (20) and $A_0(x)$ and B_0 from (15) (B is constant) we are led to the recurrence relations

$$\begin{aligned} Y_\xi &= B\left(-\xi\cot(x) + \frac{\mathrm{d}}{\mathrm{d}x}\right) Y_{\xi-1} \\ Y_{\xi-1} &= -\frac{1}{B(E - \xi^2)}\left(\xi\cot(x) + \frac{\mathrm{d}}{\mathrm{d}x}\right) Y_\xi \end{aligned} \tag{75}$$

which connect solutions for real values of ξ.

Because the constant is

$$C = B^2(E - \xi^2) \tag{76}$$

the equations above apply to square integrable functions provided that $E \geqslant \xi^2$. The choice $B = 1/\sqrt{E - \xi^2}$ leads to recurrence relations that conserve the norm of $Y_\xi(x)$:

$$\begin{aligned} Y_\xi &= \frac{1}{\sqrt{E - \xi^2}}\left(-\xi\cot(x) + \frac{\mathrm{d}}{\mathrm{d}x}\right) Y_{\xi-1} \\ Y_{\xi-1} &= -\frac{1}{\sqrt{E - \xi^2}}\left(\xi\cot(x) + \frac{\mathrm{d}}{\mathrm{d}x}\right) Y_\xi. \end{aligned} \tag{77}$$

The physical value of E is such that these recurrence relations terminate conveniently to avoid the occurrence of non-integrable functions.

The recurrence relations or ladder equations just derived are quite general. For example, if we choose $\xi = -m - \frac{1}{2}$, $m = 0, 1, \ldots$, then $\xi(\xi + 1) = m^2 - \frac{1}{4}$, and the acceptable values of E that truncate the ladder are $E = l(l+1) + \frac{1}{4}$, $l = 0, 1, \ldots$. Therefore, $\mathcal{P}_l^m = \sin(x)^{-1/2} Y_{-m-1}$ are the associated spherical harmonics [5]. If $\xi = -m - \gamma$, where γ is an arbitrary positive parameter, then $\xi(\xi+1) = (m+\gamma)(m+\gamma-1)$ and $E = (l+\gamma)^2$. In this case we obtain the generalized spherical harmonics $\mathcal{P}_{l,\gamma}^m = \sin(x)^{-\gamma} Y_{-m-\gamma}$.

8. Constructive algorithm

The point of view adopted above differs from the strategy commonly followed in both the factorization method and supersymmetric quantum mechanics. Instead of constructing exactly solvable models from a trial superpotential or a similar function we have concentrated on the question whether a given model is exactly solvable or not. In some particular cases the answer is straightforwardly provided by the simple consistency condition (21). It is not difficult to integrate this equation and generate a constructive algorithm. By means of the new functions $S = R + R_0$ and $D = R - R_0$ we rewrite (21) as $(S'/D)' = -D$, which after integration yields

$$S(x) = c_2 + c_1 \int^x D(x')\,\mathrm{d}x' - \int^x D(x') \int^{x'} D(x'')\,\mathrm{d}x''\,\mathrm{d}x' \tag{78}$$

where c_1 and c_2 are arbitrary constants. We realize that $R(x)$, $R_0(x)$ and $A(x)$ are completely determined by the choice of $D(x)$ and the integration constants c_1 and c_2. The function $A(x)$ reads

$$A(x) = \frac{BS'(x)}{2D(x)} = \frac{B}{2}\left(c_1 - \int^x D(x')\,\mathrm{d}x'\right). \tag{79}$$

The constructive algorithm then proceeds as follows: starting from a trial function $D(x)$ we obtain the pair of functions $R(x)$ and $R_0(x)$ which define two model potentials. If the latter differ only in the value of one or more parameters then we have produced a shape invariant potential. The quantization condition for the bound states follows from the constant C given by (16) with $P = 1$ and $\bar{B}' = 0$.

To illustrate the procedure we choose

$$D(x) = -\frac{2\xi}{\cos^2(x)} \qquad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{80}$$

which after a straightforward calculation leads to

$$R(x) = -c_1\xi\tan(x) - \frac{\xi(\xi+1)}{\cos^2(x)} + \frac{c_2}{2} \tag{81}$$

$$R_0(x) = -c_1\xi\tan(x) - \frac{\xi(\xi-1)}{\cos^2(x)} + \frac{c_2}{2} \tag{82}$$

$$A(x) = B\left(\xi\tan(x) + \frac{c_1}{2}\right) \tag{83}$$

$$C = B^2\left(\frac{c_2}{2} + \frac{c_1^2}{4} - \xi^2\right). \tag{84}$$

The functions $R(x)$ and $R_0(x)$ given above are not simply related by a change of the value of the parameter ξ. However, we easily obtain a shape-invariant model potential by setting $c_1 = \alpha/\xi$, α being another arbitrary real constant. Following the procedure outlined in the preceding sections one can easily derive the recurrence relations or ladder equations connecting solutions Ψ_ξ and $\Psi_{\xi-1}$ and calculate the eigenvalues $E = c_2/2$ of the potential energy function

$$V(x) = \alpha \tan(x) + \frac{\xi(\xi+1)}{\cos^2(x)} \tag{85}$$

that represents an asymmetric infinite well in the interval $(-\pi/2, \pi/2)$. For example, the eigenvalues and the ground state eigenfunction are, respectively,

$$E_n = (n+\xi+1)^2 - \frac{\alpha^2}{4(n+\xi+1)^2} \qquad n = 0, 1, \ldots \tag{86}$$

$$\Psi_0 = N \cos^{\xi+1}(x) \exp\left(-\frac{\alpha x}{2(\xi+1)}\right) \tag{87}$$

where N is a normalization factor.

In passing we mention that it is possible to derive the recurrence relations and eigenvalues for some quantum-mechanical problems from appropriate changes of variables and parameters in other models. For example, substituting $\mathrm{i}x$, $\mathrm{i}\alpha$, $-\xi - 1$ and $-E_n$ for x, α, ξ and E_n in the equations above, we obtain the corresponding expressions for the Rosen–Morse potential [13].

9. Further comments and conclusions

In principle, the connection method developed here generalizes the standard factorization method [5, 6] in at least two ways. First, it does not assume a particular dependence on the potential parameters. In fact, the functions $\tilde{R}(x)$ and $\tilde{R}_0(x)$ do not necessarily correspond to the same physical problem as shown in the applications of perturbation theory to solve the coupled differential equations for $A(x)$ and $B(x)$ in an approximate systematic way [1–4]. Second, it is not necessary to transform the differential equation into another one with $P = 1$ and $Q = 0$ because the connection method with $B' \neq 0$ applies to the general case as illustrated in appendix 2. In principle, this feature makes the connection method more general than supersymmetric quantum mechanics in which $B = 1$. The fact that in many cases it may be too difficult to solve the general differential equations for $A(x)$ and $B(x)$ without a previous transformation of the original differential equation into a simpler one does not make the method less general.

Throughout the paper we adopted two different points of view with respect to the particular case of shape-invariant potentials. First, we concentrated on whether a given potential is shape-invariant and developed a useful consistency condition to answer this question in the particular case considered by the factorization method and supersymmetric quantum mechanics, namely $P = 1$, $Q = 0$ and $B' = 0$. If two functions $R(x)$ and $R_0(x)$ differing in the value of one or more parameters satisfy the consistency condition then we have a shape-invariant potential. Second, from the consistency equation we derived closed-form expressions for the construction of pairs of potentials similar to the partner potentials in supersymmetric quantum mechanics [7, 8, 12, 13]. Although this constructive algorithm is equivalent to (and certainly can be derived from) that in supersymmetric quantum mechanics, it may nevertheless facilitate the obtention of new shape-invariant potentials.

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Appendix 1

In this appendix we show a general transformation of the second-order differential equation (1) which contains as a particular case the transformation used in section 2 to eliminate the term QY'. Although this more general transformation based on a simultaneous change of both the dependent and independent variables is well known and has been extensively used by Infeld and Hull [5] in their discussion of the factorization cases, we outline it here for the sake of completeness.

In order to cast the differential equation

$$P(x)Y''(x) + Q(x)Y'(x) + \bar{R}(x)Y(x) = 0 \tag{A1.1}$$

into a more convenient form for the application of either the factorization method or supersymmetric quantum mechanics we change the dependent and independent variables according to

$$Y(x) = F(z)\Psi(z) \qquad x = x(z). \tag{A1.2}$$

From now on a prime on x indicates a derivative with respect to z and a prime on P or Q denotes a derivative with respect to x. Choosing

$$F = (x')^{1/2} \exp\left(- \int \frac{Q(x)}{2P(x)} \, \mathrm{d}x \right) \tag{A1.3}$$

we eliminate the term QY' and the differential equation (A1.1) becomes

$$\frac{\mathrm{d}^2\Psi(z)}{\mathrm{d}z^2} + R(z)\Psi(z) = 0 \tag{A1.4}$$

where

$$R = (x')^2 \left\{ \frac{\bar{R}}{P} + \frac{x'''}{2(x')^3} - \frac{3(x'')^2}{4x'} - \left(\frac{Q}{2P}\right)' - \frac{Q^2}{4P^2} \right\}. \tag{A1.5}$$

Notice that (A1.3) and (A1.5) reduce to (4) and (7), respectively, in the particular case $x(z) = z$. By means of this transformation one can sometimes transform an equation which does not satisfy the consistency condition (21) into another equation that satisfies it.

As an example consider the differential equation

$$(1 - x^2)Y''(x) - \gamma x Y'(x) + \lambda Y(x) = 0. \tag{A1.6}$$

Setting $P = 1 - x^2$, $Q = -\gamma x$, $\bar{R} = \lambda$ and $x(z) = -\cos(z)$ in (A1.5) we obtain an inverse quadratic potential

$$R(z) = \lambda + \frac{(\gamma - 1)^2}{4} - \frac{(\gamma - 1)(\gamma - 3)}{4\sin^2(z)} \tag{A1.7}$$

that satisfies the consistency equation as shown in section 7. When $\gamma = 2m+3$ the function (A1.7) becomes the expression derived by Infeld and Hull from the differential equation for the Gegenbauer functions [5].

Appendix 2

Here we show that the connection method introduced in section 2 is equivalent to the factorization of second-order differential equations of the form (1) that are more general than those customarily treated by both the standard factorization method and supersymmetric quantum mechanics [5–8]. Instead of removing the terms QY' and QZ' in (1) and (2) as we did in section 2, here we directly write

$$Y(x) = \left(A(x) + \bar{B}(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) Z(x) \tag{A2.1}$$

$$Z(x) = \left(A_0(x) + \bar{B}_0(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) Y(x). \tag{A2.2}$$

Substituting (A2.2) into (A2.1) and re-ordering the resulting equation conveniently, we have

$$\bar{B}_0\bar{B}Y'' + (A\bar{B}_0 + A_0\bar{B} + \bar{B}\bar{B}_0')Y' + (AA_0 + \bar{B}A_0' - 1)Y = 0. \tag{A2.3}$$

Comparison of this equation with (1) suggests the choices

$$\bar{B}_0\bar{B} = GP \tag{A2.4}$$

$$A\bar{B}_0 + A_0\bar{B} + \bar{B}\bar{B}_0' = GQ \tag{A2.5}$$

$$AA_0 + \bar{B}A_0' - 1 = G\bar{R} \tag{A2.6}$$

where G is an arbitrary function of x. Analogously, substitution of (A2.1) into (A2.2) leads to

$$A\bar{B}_0 + A_0\bar{B} + \bar{B}_0\bar{B}' = GQ \tag{A2.7}$$

$$AA_0 + \bar{B}_0A' - 1 = G\bar{R}_0 \tag{A2.8}$$

and also to (A2.4), which tells us that the unknown function G is the same in both cases.

Subtracting (A2.7) from (A2.5) yields $\bar{B}\bar{B}_0' - \bar{B}_0\bar{B}' = 0$, which shows that $\bar{B}_0$ is proportional to $\bar{B}$:

$$\bar{B}_0 = -\frac{\bar{B}}{C} \tag{A2.9}$$

where C is the proportionality constant. The unknown function G follows from (A2.4) and (A2.9): $G = -\bar{B}^2/CP$. Adding (A2.5) and (A2.7) and using (A2.4) and (A2.9) enables us to obtain

$$A_0 = \frac{1}{C}\left(A + \bar{B}' - \frac{Q}{P}\bar{B}\right). \tag{A2.10}$$

Substitution of all those results into (A2.8) yields the constant C:

$$C = A^2 + A\bar{B}' - \bar{B}A' - \frac{Q}{P}A\bar{B} + \frac{\bar{B}^2}{P}R_0. \tag{A2.11}$$

In order to obtain a differential equation for $\bar{B}$ we subtract (A2.8) from (A2.6) and use the results above:

$$\bar{B}'' - \left(\frac{Q\bar{B}}{P}\right)' + (\bar{R} - \bar{R}_0)\frac{\bar{B}}{P} + 2A' = 0. \tag{A2.12}$$

The differential equation for A follows from differentiation of (A2.11) with respect to x:

$$A'' + (\bar{R} - \bar{R}_0)\frac{A}{P} + \frac{Q}{P}A' - \frac{2\bar{R}_0}{P}\bar{B}' - \left(\frac{\bar{R}_0}{P}\right)'\bar{B} = 0. \tag{A2.13}$$

The expressions just derived generalize those in section 2 and enable one to treat eigenvalue equations of the form (1) and (2) directly without their previous transformation into more tractable forms. The problem thus reduces to solving the differential equations for $A(x)$ and $B(x)$ (or $\bar{B}(x) = P(x)B(x)$).

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